smaller particles, as has already been suggested by J. J. Thomson from entirely different evidence; but the results are too few to make further speculation on their meaning of much value.

"Note on the Complete Scheme of Electrodynamic Equations of a Moving Material Medium, and on Electrostriction."

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This note forms a supplement to my third memoir on the "Dynamical Theory of the Æther,"* to the sections of which the references are made.

1. It is intended in the first place to express with full generality the electrodynamic equations of a material medium moving in any manner, thus completing the scheme which has been already developed subject to simplifying restrictions in the memoirs referred to. To obtain a definite and consistent theoretical basis it was necessary to contemplate the material system as made up of discrete molecules, involving in their constitutions orbital systems of electrons, and moving through the practically stagnant æther. It is unnecessary, for the mere development of the equations, to form any notion of how such translation across the æther can be intelligibly conceived: but, inasmuch as its strangeness, when viewed in the light of motion of bodies through a material medium and the disturbance of the medium thereby produced, has often led to a feeling of its impossibility, and to an attitude of agnosticism with reference to æthereal constitution, it seems desirable that a kinematic scheme such as was there explained, depending on the conception of a rotationally elastic æther, should have a place in the foundations of æther-theory. Any hesitation, resting on *à priori* scruples, in accepting as a working basis such a rotational scheme, seems to be no more warranted than would be a diffidence in assuming the atmosphere to be a continuous elastic medium in treating of the theory of sound. It is known that the origin of the elasticity of the atmosphere is something wholly different from the primitive notion of statical spring, being in fact the abrupt collisions of molecules: in the same way the rotational quality of the incompressible æther, which forms a sufficient picture of its effective constitution, may have its origin in something more fundamental that has not yet even been conceived. But in each case what is important for immediate practical purposes is a condensed and definite basis from which to develop the interlacing ramifications of a physical scheme: and in each case this is obtained by the use of a representation which a deeper knowledge may after-

* 'Phil. Trans.,' A (1897).

wards expand, transform, and even modify in detail. Although, however, it is possible that we may thus be able ultimately to probe deeper into the problem of æthereal constitution, just as the kinetic theory has done in the case of atmospheric constitution, yet there does not seem to be at present any indication whatever of any faculty which can bring that medium so near to us in detail as our senses bring the phenomena of matter: so that from this standpoint there is much to be said in favour of definitely regarding the scheme of a continuous rotationally elastic æther as an ultimate one.

A formal scheme of the dynamical relations of free æther being postulated after the manner of Maxwell and MacCullagh, and a notion as clear as possible obtained of the æthereal constitution of a molecule and its associated revolving electrons, by aid of the rotational hypothesis, it remains to effect with complete generality the transition between a molecular theory of the æthereal or electric field which considers the molecules separately, and a continuous theory expressed by differential equations which take cognizance only of the properties of the element of volume, the latter alone being the proper domain of mechanical as distinct from molecular theory. This transformation is, as usual, accomplished by replacing summations over the distribution of molecules by continuous integrations over the space occupied by them. In cases where the integrals concerned all remain finite when the origin to which they refer is inside the matter so that the lower limit of the radius vector is null, there is no difficnlty in the transition: this is for example the case with the ordinary theory of gravitational forces. But in important branches of the electric theory of polarised media, some of the integral expressions become infinite under these circumstances; and this is an indication that it is not legitimate to replace the effect of the part of the discrete distribution of molecules which is adjacent to the point considered by that of a continuous material distribution. The result of the integration still, however, gives a valid estimate of the effect of the material system *as a whole*, if we bear in mind that the infinite term coming in at the inner limit really represents a finite part of the result depending *solely* on the local molecular configuration, a part whose actual magnitude could be determined only when that configuration is exactly assigned or known. The consideration of this indeterminate part is altogether evaded by means of a general mechanical principle which I have called the principle of mutual compensation of molecular forcives. This asserts that in such cases, when a finite portion of the effect on a molecule arises from the action of the neighbouring molecules, this part must be omitted from the account in estimating the *mechanical* effect on an element of volume of the medium; indeed otherwise mechanical theory would be impossible. The mutual, statically equilibrating,

actions of *adjacent* molecules determine the structure of the medium, and any change therein involves change in its *local* physical constants and properties, which may or may not be important according to circumstances: but such local action contributes nothing towards polarising or straining the element of mass whose structure is thus constituted, and therefore nothing to mechanical excitation, unless at a place where there is abrupt change of density.* In the memoir above mentioned this molecular principle was applied mainly to determine the mechanical stress in a polarised material medium. It necessarily also enters into the determination of the electrodynamic equations of a moving medium treated as a continuous system, and even of a magnetised medium at rest, from consideration of its molecular constitution. It is here intended only to record in precise form the general scheme that results from it, details of demonstration being for the present reserved. Everything being expressed in a continuous scheme per unit volume, let (u', v', w') denote the current of conduction, (u, v, w) the total current of Maxwell, (f, g, h) the electric displacement in the æther and (f', g', h') the electric polarisation of the molecules so that the total so-called displacement flux of Maxwell is $(f+f',\ g+g',\ h+h')$; let ρ be the volume-density of uncompensated electrons or the density of free charge, let (A, B, C) be the magnetisation, and (p, q, r) the velocity of the matter with respect to the stagnant æther. As before explained (§ 13, footnote), the convection of the material polarisation (f', g', h') produces a *quasi*-magnetisation $(rg'-qh',\ ph'-rf',\ qf'-pg')$ which adds on to (A, B, C). Also, as before shown, the vector potential of the æthereal field, so far as it comes from the molecular electric whirls which constitute magnetisation, is given, for a point outside the magnetism, by

$$\mathrm{F} = \int\left(\mathrm{B}\frac{d}{dz} - \mathrm{C}\frac{d}{dy}\right)\frac{1}{r}\,d\tau$$

$$= \int(\mathrm{B}n - \mathrm{C}m)r^{-1}d\mathrm{S} + \int\left(\frac{d\mathrm{C}}{dy} - \frac{d\mathrm{B}}{dz}\right)\frac{1}{r}\,d\tau,$$

(lmn) being the direction vector of $\delta\mathrm{S}$, and therefore is that due to a bodily current system $\left(\frac{d\mathrm{C}}{dy} - \frac{d\mathrm{B}}{dz}, \ldots, \ldots\right)$ together with current sheets on the interfaces. When the point is inside the magnetism, there are still no infinities in the integral expressing F, and this transformation of it by partial integration is still legitimate. But

* This exception explains why the mechanical tractions on an interface, determined in § 36 as the limit of a gradual transition, are different from the forces on the Poisson equivalent interfacial distribution.

the spacial differential coefficients of (F, G, H) are also involved in the forcives of the æthereal field, and with them the case is different: the transformation by parts is then analytically wrong, owing to neglect of the infinite elements at the origin, while in actuality a finite portion of the whole effect arises from the influence of the neighbouring molecules. We have, therefore, by the molecular principle, to separate the infinite elements from the integrals and leave them out of account; and this is effected by employing the second form above for F, which differs from the first form only in having got rid of the local terms at the origin in its differential coefficients. Thus it is not merely convenient, but even necessary for a mechanical theory, which considers distributions instead of individual molecules, to replace magnetism by its equivalent continuous current system as here. The *quasi*-magnetism arising from electric convection adds to this equivalent current system the additional bodily terms

$$\left\{\frac{d}{dy}(qf'-pg')-\frac{d}{dz}(ph'-rf'),\ \ldots\ldots,\ \ldots\ldots\right\}$$

together with surface sheets: thus the volume current so added has for x-component

$$\frac{\delta f'}{dt}-\frac{df'}{dt}-p\left(\frac{df'}{dx}+\frac{dg'}{dy}+\frac{dh'}{dz}\right)-f'\frac{dp}{dx}-g'\frac{dp}{dy}-h'\frac{dp}{dz},$$

where $\frac{\delta f}{dt}$ represents $\frac{df'}{dt}+\frac{dpf'}{dx}+\frac{dqf'}{dy}+\frac{drf'}{dz}$, or the rate of change of f' supposed associated with the moving matter. Combining all these parts, the current and magnetism together are completely represented as regards determination of electric effect by what we may call the *total effective current* (u_1, v_1, w_1) where

$$u_1=u'+\frac{dC}{dy}-\frac{dB}{dz}+\frac{df}{dt}+\frac{\delta f'}{dt}-\left(\frac{dpf'}{dx}+\frac{dpg'}{dy}+\frac{dph'}{dz}\right)+p\rho,$$

together with superficial current sheets arising from the true magnetism (A, B, C) and the electric convection. Since ρ is equal to

$$\frac{d(f+f')}{dx}+\frac{d(g+g')}{dy}+\frac{d(h+h')}{dz}$$

we may write

$$u_1=u'+\frac{dC}{dy}-\frac{dB}{dz}+\frac{df}{dt}+\frac{\delta f'}{dt}-\left(f'\frac{dp}{dx}+g'\frac{dp}{dy}+h'\frac{dp}{dz}\right)+p\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)$$

in which the last term may be expressed as $-\nabla^2\Psi/4\pi c^2$.

It is to be observed that this effective current satisfies the condition of incompressible flow,* which by definition (or rather by the æthereal constitution) is necessarily satisfied by the *total current* (u, v, w) of the previous memoirs; for the additional terms which represent the magnetism clearly satisfy the stream relation.† The remainder of the scheme of electrodynamic relations is established as in the previous memoirs. Thus (F, G, H) now representing simply $\int(u_1, v_1, w_1)r^{-1}d\tau$, which satisfies the stream relation $d\mathrm{F}/dx+d\mathrm{G}/dy+d\mathrm{H}/dz=0$ because (u_1, v_1, w_1) is a stream vector, we deduce an electric force (P, Q, R) acting on the electrons, where

$$\mathrm{P} = cq-br-d\mathrm{F}/dt-d\Psi/dx,$$

also an æthereal force (P′, Q′, R′) straining the æther, where

$$\mathrm{P}' = (4\pi \mathrm{C}^2)^{-1}f = -d\mathrm{F}/dt-d\Psi/dx,$$

the function Ψ being determined in each problem so as to avoid æthereal compression.

Across an abrupt transition, F, G, H and the normal component of (u_1, v_1, w_1) must be continuous, thus making up the *four* necessary and sufficient interfacial conditions. The gradients of F, G, H are, however, not continuous when there is magnetisation or dielectric convection, on account of the effective interfacial current sheets before mentioned.

The exact value of the mechanical force (X, Y, Z) per unit volume, comes out as

$$\mathrm{X} = \left(v-\frac{dg}{dt}\right)\gamma-\left(w-\frac{dh}{dt}\right)\beta+\mathrm{A}\frac{d\alpha}{dx}+\mathrm{B}\frac{d\alpha}{dy}+\mathrm{C}\frac{d\alpha}{dz}$$
$$+f'\frac{d\mathrm{P}'}{dx}+g'\frac{d\mathrm{P}}{dy}+h'\frac{d\mathrm{P}'}{dz}+\rho\mathrm{P}',$$

where $\alpha = d\mathrm{H}/dy-d\mathrm{G}/dz-4\pi\mathrm{A}$.

* It is proposed to call a flow-vector which obeys this condition a *stream*, the more general term *flow* or *flux* including cases like the variable stage of the flow of heat in which the condition of absence of convergence is not satisfied. The two main classes of physical vectors may be called *fluxes* and *gradients*, the latter name including such entities as forces and being especially appropriate when the force is the gradient of a potential. Lord Kelvin's term *circuital* flux has previously been used to denote a *stream* vector; but it is perhaps better to extend it to a general vector which is directed along a system of complete circuits.

† The (u, v, w) of § 13, however, included a part arising from convection of electric polarisation. Notice that when this is transferred to the magnetism, as here, we have $u = u'+df/dt+df'/dt+p\rho$: thus when there is no conduction and ρ is therefore wholly convected so that $\delta\rho/dt$ is null, the stream character of the total current simply requires $d(f+f')/dx+d(g+g')/dy+d(h+h')/dz = \rho$, so that the formulation is now easier and more natural.

In these formulæ, with the exception of the one for (u_1, v_1, w_1) above, (A, B, C) includes the *quasi*-magnetism arising from electric convection, while (u, v, w) is the total electric current that remains after all magnetic effect of whatever type has been omitted. It is to be noted that the final terms in **X** involve in strictness the æthereal force, instead of the electric force as in § 39.

It follows from the formula for (P, Q, R) that

$$\frac{d\mathrm{R}}{dy}-\frac{d\mathrm{Q}}{dz} = -\frac{\delta a}{dt}+\left(a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}\right)p\,;$$

hence *Faraday's circuital relation holds good provided the velocity* (p, q, r) *of the matter is uniform in direction and magnitude.*

Again, since (F, G, H) is a stream vector,

$$\frac{dc}{dy}-\frac{db}{dz} = -\nabla^2\mathrm{F} = 4\pi\left(u+\frac{d\mathrm{C}}{dy}-\frac{d\mathrm{B}}{dz}\right),$$

where (u, v, w) represents the total current of Maxwell, and (A, B, C) the whole of the magnetism and the *quasi*-magnetism of convection: hence

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz} = 4\pi u,$$

so that *Ampère's circuital relation holds, with the above definition of* (α, β, γ), *under all circumstances.*

But in circumstances of electric convection these two circuital relations would not usually by themselves form the basis of a complete scheme of equations, as they do when the material medium is at rest.

To complete the scheme, the above dynamical equations must be supplemented by the observational relations connecting the conduction current with the electric force, the electric polarisation with the electric force, and the magnetism with the magnetic force. In the simplest case of isotropy these relations are of types

$$u' = \sigma\mathrm{P}, \quad f' = (\mathrm{K}-1)/4\pi\mathrm{C}^2\mathrm{P}, \quad \mathrm{A} = \kappa\alpha+(rg'-qh').$$

It is to be observed that the physical constants which enter into the expression of these relations will presumably be altered by motion through the æther of the material system to which they belong: but because there is nothing unilateral in the system, a reversal of this motion should not change the constants, therefore their alteration must depend on the square of the ratio of the velocity of the system to that of radiation, and would only enter in a second approximation.

The various problems relating to electric convection and optical

aberration worked out in §§ 14—16, pp. 225—229, will be found to fit into this scheme. I take the opportunity of correcting an erratum in p. 226, lines 16, 17, which should read

$$\Psi_1 = \tfrac{1}{3}(1+K^{-1})wcr^2 + Ar^2(\cos^2\theta - \tfrac{1}{3}) + A'$$

$$\Psi_2 = Br^{-3}(\cos^2\theta - \tfrac{1}{3}) + B'r^{-1},$$

with of course different values of the constants.

2. In a material dielectric the bodily mechanical forcive is derived from a potential $-(K-1)F^2/8\pi$, and there is also a normal inward traction $KF^2/8\pi$ where it abuts on conductors. For the thin dielectric shell of a condenser this forcive could be balanced by a hydrostatic pressure $(K-1)F^2/8\pi$ together with a Maxwell stress consisting of a pressure $F^2/8\pi$ along the lines of force and an equal tension at right angles to them: in fact this reacting system gives the correct traction over the faces of the sheet and the correct forcive throughout its substance. If the sheet has an open edge the tractions on that edge are however not here attended to; when the sheet is thin these are of small amount, and their effect is usually local, as otherwise the nature of the edge would be an important element. Moreover, in the most important applications of the formula the edge is of small extent, so that they form a local statically balanced system. The stress above specified will thus represent the material elastic reaction, provided the strains in the different elements of volume, which correspond to it, can fit together without breach of continuity of the solid material. This condition will be secured if the shell is of uniform thickness so that F is constant all over it: in that case, therefore, the elastic reaction in the material will make up a pressure $KF^2/8\pi$ along the lines of force and a pressure $(K-2)F^2/8\pi$ in all directions at right angles to them, which is the result obtained for solids in § 76.

If, however, the coatings of the condenser are not supported by the dielectric shell, the elastic reaction in the shell will be simply a pressure $(K-1)F^2/8\pi$ uniform in all directions. This is what actually occurs in the case of a fluid dielectric, where such support is not mechanically possible.

It appeared from § 79 that in glass there is actually an increase of volume under electric excitation, while the mechanical forces would produce a diminution: and the same is true for most dielectric liquids, the fatty oils being exceptions,* though by a confusion between action and reaction the result was there stated as the opposite. It thus appears that in general an intrinsic expansion, in addition to the effects of the mechanical force, accompanies electric

* In the cognate case of magnetisation of ferrous sulphate solution, Hurmuzescu finds a contraction of volume.

excitation of material dielectrics. This circumstance will perhaps recall to mind Osborne Reynolds' theory of the dilatancy of granular media, which explains that the discrete elements of such media tend to settle down *under the mutual influences of their neighbours* so as to occupy the smallest volume, and therefore any disturbing cause has a tendency to increase the volume.

In § 80, on the influence of electric polarisation on ripple velocity, the result stated for dielectrics should be doubled. It is to be remarked that a horizontal dielectric liquid surface becomes unstable in a uniform vertical electric field when the square of the total continuous vertical electric displacement exceeds the moderate value $\frac{K_1K_2(K_2+K_1)}{2\pi(K_2-K_1)^2}\{(\rho_2-\rho_1)gT\}^{\frac{1}{2}}$ electrostatic units, T being the capillary tension. For a conducting liquid instability ensues when the square of the surface electric density exceeds $\frac{K_1^2}{2\pi}\{\rho_2-\rho_1)gT\}^{\frac{1}{2}}$ electrostatic units. In exciting a dielectric liquid by the approach of an electrified rod it must often have been noticed that when the rod is brought too near, the liquid spurts out vigorously in extremely fine filaments or jets: the fineness of the filaments may be explained, in part at any rate, after Lord Rayleigh ('Phil. Mag.,' 1882, "Theory of Sound," § 364), without assuming an escape of electricity into the liquid, as arising from the circumstance that it is only narrow crispations of the surface, and not extensive deformations, that become unstable.

The opportunity is taken to correct other errata in the memoir, 'Phil. Trans.,' A, 1897, as follows:—

page 252, line 15, *read* 8π *for* 4π.
,, 253, ,, 18, the factor C^2 is omitted.
,, 297, ,, 4, *dele* m.